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SOME INVERSE SCATTERING PROBLEMS FOR TWO-DIMENSIONAL SCHRÖDINGER OPERATOR

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Abstract - We prove that in two-dimensional potential scattering the leading order singularities (in some special cases - all singularities) of unknown potential are obtained exactly by the Born approximation. The proof is based on new estimates for the continuous spectrum of the Laplacian and new estimates for the Green-Faddeev's function in the weighted L^p-spaces. Using these estimates, we prove the well-known Saito's formula, an uniqueness theorem for the reconstruction of the unknown potential by the scattering amplitude and recovering singularities in general scattering, in backscattering, in fixed angle scattering and in fixed energy scattering. We prove also a new asymptotical formula for the Fourier transform of the unknown potential.

1. INTRODUCTION

Let $H = -\Delta + q(x)$ be the Schrödinger operator in R^2 with the real-valued potential $q(x)$. We assume that the potential belongs to the weighted space $L^p_\sigma(R^2)$ defined by the norm

$$\|q\|_{p,\sigma} = \left(\int_{R^2} (1+|x|)^{p\sigma} |q(x)|^p \, dx \right)^{1/p}, \tag{1}$$

where $1 < p \leq \infty$ and σ is a nonnegative number that will be specified later.

Below we use the following notations. The space H^t denotes the usual L^2- based Sobolev space and the space W^t_p denotes the L^p- based Sobolev space in R^2.

Under the above assumptions on the potential the Hamiltonian H is a self-adjoint operator in $L^2(R^2)$. The spectrum of this operator consists of a continuous spectrum, filling out the positive real axis (with possible positive eigenvalues), and a possible negative discrete spectrum of finite multiplicity with zero as the only possible accumulation point. In addition we suppose the potential has some power decay at the infinity

$$|q(x)| \leq C|x|^{-\mu} \tag{2}$$

for $|x|$ large and for some $\mu > 2$. Then the discrete spectrum (if it exists) is purely negative and finite, there are no positive eigenvalues and zero belongs to the continuous spectrum $[0, \infty)$ (see [2],[18]). In this case we can define for arbitrary $k \in R, k \neq 0$, the scattering solutions of the homogeneous Schrödinger equation

$$(H - k^2)u(x,k) = 0$$

to be the unique solutions of the Lippmann-Schwinger equation

$$u(x,k,\vartheta) = e^{ik(x,\vartheta)} - \int_{R^2} G^+_k(|x-y|)q(y)u(y,k,\vartheta)\, dy,$$

where $\vartheta \in S^1$ and the outgoing fundamental solution of the Helmholtz equation G^+_k is defined as

$$G^+_k(|x|) = \frac{i}{4} H^{(1)}_0(|k||x|),$$

where $H^{(1)}_0$ is the Hankel function of the first kind and 0 order. Recall that the function G^+_k is the kernel of the integral operator $(-\Delta - k^2 - i0)^{-1}$.

The solutions $u(x,k,\vartheta)$ for $k > 0$ admit asymptotically, as $|x| \to +\infty$ uniformly with respect to ϑ, a representation

$$u(x,k,\vartheta) = e^{ik(x,\vartheta)} + \frac{1+i}{4\sqrt{\pi}} e^{ik|x|} k^{-1/2} |x|^{-1/2} A(k,\vartheta',\vartheta) + o\left(\frac{1}{|x|^{1/2}}\right),$$

where $\vartheta' = \frac{x}{|x|} \in S^1$ and the function $A(k,\vartheta',\vartheta)$ is called a scattering amplitude defined as

$$A(k,\vartheta',\vartheta) = \int\limits_{R^2} e^{-ik(\vartheta',y)} q(y) u(y,k,\vartheta)\, dy. \tag{3}$$

It what follows we extend A to negative k by $A(k,\vartheta',\vartheta) = \overline{A(-k,\vartheta',\vartheta)}$ to obtain a well-defined scattering amplitude for all $k \in R, k \neq 0$. We use also the fact that $A(k,\vartheta',\vartheta) = A(k,-\vartheta,-\vartheta')$.

We will consider the problem of recovering the singularities of the potential and the potential itself assuming that we know the scattering amplitude $A(k,\vartheta',\vartheta)$ for certain data.

As a different data for the reconstruction of unknown potential $q(x)$ we consider the kernel $G_q(x,y,k)$ of the integral operator $(H - k^2 - i0)^{-1}$ which is the solution of the following integral equation:

$$G_q(x,y,k) = G_k^+(|x-y|) - \int\limits_{R^2} G_k^+(|x-z|) q(z) G_q(z,y,k)\, dz. \tag{4}$$

Definition 1 *We say that the Hamiltonian H has a resonance at zero if the homogeneous Lippmann-Schwinger equation for $k=0$*

$$v(x) = -\int\limits_{R^2} G_0^+(|x-y|) q(y) v(y)\, dy,$$

where $G_0^+(|x|) = \frac{1}{2\pi} \log \frac{1}{|x|} + c_0$ and c_0 is known positive constant, has a nontrivial continuous solution vanishing uniformly at the infinity.

As it follows from [20] and [21] if the Hamiltonian has no resonance at zero then the scattering amplitude can be well-defined for $k=0$ by continuity and it is equal to zero in this case. Therefore, in this case the inverse scattering problem with zero energy makes no sense.

It follows from (3) that for every fixed point $\xi \in R^2$

$$(Fq)(\xi) = \lim_{k \to +\infty} A(k,\vartheta',\vartheta), \quad \xi = k(\vartheta - \vartheta')$$

and also we have

$$(Fq)(2\xi) = A(k,-\vartheta,\vartheta) + o_k(1), \quad k = |\xi|, \quad \vartheta = \frac{\xi}{|\xi|},$$

where F is the ordinary Fourier transform in R^2. The latter formulas justify the following definitions.

Definition 2 *The inverse Born approximation $q_B(x)$ of the potential $q(x)$ is defined as follows:*

$$q_B(x) := \frac{1}{32\pi^3} \int\limits_{R\times S^1 \times S^1} e^{-ik(\vartheta - \vartheta',x)} A(k,\vartheta',\vartheta) |k| |\vartheta - \vartheta'|^2\, dk\, d\vartheta\, d\vartheta'. \tag{5}$$

Definition 3 *The inverse Born backscattering approximation $q_B^b(x)$ of the potential $q(x)$ is defined as follows:*

$$q_B^b(x) := \frac{1}{4\pi^2} \int\limits_{R^2} e^{-i(\xi,x)} A\left(\frac{|\xi|}{2}, -\frac{\xi}{|\xi|}, \frac{\xi}{|\xi|}\right) d\xi. \tag{6}$$

Definition 4 *The inverse Born fixed angle scattering approximation $q_B^{\vartheta_0}(x)$ of the potential $q(x)$ is defined as follows:*

$$q_B^{\vartheta_0}(x) := \frac{1}{16\pi^2} \int\limits_{R\times S^1} e^{-ik(\vartheta - \vartheta_0,x)} A(k,\vartheta_0,\vartheta) |k| |\vartheta - \vartheta_0|^2\, dk\, d\vartheta, \tag{7}$$

where $\vartheta_0 \in S^1$ is fixed.

It is very easy to see that within the Born approximation, the scattering amplitude is simply the Fourier transform of the unknown potential. The weaker the potential, the better is this approximation. But even when the potential is not weak the Fourier transform of a scattering amplitude contains essential information of the potential as was shown in [14], [15] and [22] in two dimensions. In a series of papers starting from 1969 Prosser [17] has shown that the inverse backscattering problem has a unique solution assuming the potential has a small enough weighted Hölder norm. Generic uniqueness results for the backscattering problem in two and three dimensions were obtained by Eskin and Ralston [4]-[6]: they proved that the nonlinear operator taking the potential to the backscattering data is an analytic local homomorphism on an open and dense set of an appropriate functional space (some subsets of Sobolev spaces). Without any smallness assumptions Stefanov [23] has shown that if two compactly supported $L^{\infty}-$ potentials q_1 and q_2 have the same backscattering data and in addition he has assumed $q_1 \geq q_2$, then in fact $q_1 = q_2$. Stefanov [23]-[24] gave also a simple proof in three dimensions for the generic uniqueness of the backscattering problem and the fixed angle scattering problem in the case of compactly supported potentials belonging to Sobolev space W^4_{∞}. We recall that in the case of less singularity of the potentials (compare with our case in the present paper) Sun and Uhlmann [27] -[28] have considered related problems in two dimensions with the fixed energy data. We have to mention here the articles of Novikov [12] and Novikov and Henkin [13] where some similar problems for singular potentials were considered and also the article of Ruiz [19] about the reconstruction of singularities in the fixed angle scattering problem in two-dimensional case for the potentials with compact support from Sobolev space $H^s(R^2)$ for some non-negative s.

The following estimates for the resolvent of the Laplacian (see [16]) on the continuous spectrum play the key role in this work:

$$\|(-\Delta - k^2 - i0)^{-1} f\|_{\frac{2p}{p-1},-\delta} \leq \frac{C}{|k|^{\gamma}} \|f\|_{\frac{2p}{p+1},\delta}, \tag{8}$$

where $\gamma = 2 - 2/p$ and $\delta = 0$ for $1 < p \leq 3/2$ and $\gamma = 1 - 1/2p$ and $\delta > 1/2 - 3/4p$ for $3/2 < p \leq \infty$. This estimate follows by interpolation from the well-known results of Agmon [1] and Kenig *et al.* [8].

We need also the norm estimate of the integral operator $\mathbf{K}$ in $L^2(R^2)$

$$\mathbf{K} := |q|^{1/2}(-\Delta - k^2 - i0)^{-1}|q|^{-1/2}q, \quad q(x) \in L^p_{2\delta}(R^2).$$

The result easily follows from (8):

$$\|\mathbf{K}\|_{L^2 \to L^2} \leq \frac{C}{|k|^{\gamma}}. \tag{9}$$

with the same γ and δ.

For the inverse problem at fixed energy the crucial role is played by the following Green-Faddeev's function:

$$G_z(x) := \frac{1}{4\pi^2} \int\limits_{R^2} \frac{e^{i(x,z+\xi)}}{\xi^2 + 2(z,\xi)}\, d\xi, \tag{10}$$

where $z = (z_1, z_2) \in \mathbf{C}^2$ is a two dimensional complex vector and $z_1^2 + z_2^2 = 0$. It was proved in [16] that for $|z|$ large enough

$$\|e^{-i(x,z)} G_z * f\|_{s',-\delta} \leq \frac{C}{|z|^{\gamma}} \|f\|_{s,\delta}, \tag{11}$$

where $1 < s \leq 2$, $1/s + 1/s' = 1$, $\gamma = 4 - 4/s$ and $\delta = 0$ for $1 < s \leq 6/5$ and $\gamma = 3/2 - 1/s$ and $\delta = 5/4 - 3/2s$ for $6/5 < s \leq 2$. This esitmate follows easily by interpolation from the well-known results of Sylvester and Uhlmann [25], Kenig *et al.* [8] and Chanillo [3].

Due to the estimate (11) we can prove that there exists a special solution to the Schrödinger equation:

$$(-\Delta + q(x))u(x) = 0$$

of the form (these are the "non-physical" exponentially growing solutions or the Faddeev's solution [7]):

$$u(x, z) = e^{i(x,z)}(1 + R(x, z)). \tag{12}$$

More precisely, for a special solution of the form (12) the following result holds. Assume that the potential $q(x)$ belongs to $L^p_{2\delta}(R^2) \cap L^1(R^2)$, $1 < p \leq \infty$, with $\delta = 0$ for $1 < p \leq 3/2$ and $\delta = 1/2 - 3/4p$ for $3/2 < p \leq \infty$. Then there is $C_0 > 0$ such that for all $z, |z| > C_0$, there exists a solution to the homogeneous Schrödinger equation of the form (12) with the function $R(x, z)$ which satisfies for $|z| > C_0$ the estimate

$$\|R\|_{\frac{2p}{p-1},-\delta} \leq \frac{C}{|z|^{\gamma}}, \tag{13}$$

where $\gamma = 2 - 2/p$ for $1 < p \leq 3/2$, and $\gamma = 1 - 1/2p$ for $3/2 < p \leq \infty$, and the positive constant C does not depend on z.

The most important fact here is that the knowledge of the scattering amplitude at fixed positive energy uniquely determines, as a function of $\xi \in R^2$, the following function (see [9] and [12])

$$T_q(\xi) := \int\limits_{R^2} e^{-i(x,\xi)} q(x)(1 + R(x,z))\, dx, \quad |\xi| > \sqrt{2}C_0, \quad T_q(\xi) := 0, \quad |\xi| < \sqrt{2}C_0, \tag{14}$$

where $z = -\frac{1}{2}(\xi - iJ\xi)$ and $J = \|a_{jl}\|$ is the matrix with $a_{11} = a_{22} = 0, a_{12} = -a_{21} = 1$. In that case the Born approximation has to be in the following form.

Definition 5 *The inverse Born fixed energy approximation $q_B^f(x)$ of the potential $q(x)$ is defined as follows:*

$$q_B^f(x) := F^{-1}(T_q(\xi)), \tag{15}$$

where F^{-1} is the inverse Fourier transform of the function T_q from (14).

The fixed energy inverse problem is well-understood in dimensions higher than two (see [9], [11], [12] and [13]). The main result is that the scattering amplitude with a fixed positive energy uniquely determines a compactly supported bounded potential. However, the problem is not solved in the dimension two (see [10] and [29]). In the articles [27]-[28] Sun and Uhlmann proved that the knowledge of the scattering amplitude at fixed positive energy determines the location of the singularity as well as the jumps across the curve of the discontinuity for a compactly supported bounded potential and the reconstruction of singularities for the potentials from $L^p_{comp}(R^2)$ for $p > 2$. We improve these results for the potentials with stronger singularities.

The main idea of our considerations consists in the asymptotic expansion for the Born's potential (for all inverse problems which are presented here) in the form:

$$q_B = \sum_{j=0}^{\infty} q_j$$

analogously to the symbol expansions in the pseudodifferential calculus. By noting that q_0 is equal to the true potential $q(x)$ the problem is reduced to estimating the smoothness of the higher order terms in the Born expansion. This expansion is merely the series of iterations of the Lippmann-Schwinger equation. To prove the convergence of this series we apply estimate (9) and (11).

We are now in the position to formulate our main results about recovering of the singularities for the Schrödinger operator with singular potentials.

2. MAIN RESULTS

Theorem 1 *Assume that the potential $q(x)$ has bounded support and belongs to the space $H^s(R^2)$ for some $0 < s \leq 1$. Then $q_B(x) - q(x)$ is a bounded continuous function.*

Theorem 2 *Assume that the potential $q(x)$ has bounded support and belongs to the space $H^s(R^2)$ for some $0 < s \leq 1$. Then*

$$q_B^b(x) - q(x) - q_1(x) \in H^t(R^2),$$

where $t < \frac{s+1}{2}$ and the first nonlinear term $q_1(x)$ is a continuous function for $1/2 < s \leq 1$ and belongs to the space $H^{2s}(R^2)$ for $0 < s \leq 1/2$.

From these theorems we obtain the immediate corollary.

Corollary 1 *If a piecewise smooth compactly supported potential $q(x)$ contains the jumps over a smooth curve, then the curve and height function of the jumps are uniquely determined by the general scattering data and the backscattering data as well. Especially, for the potential being the characteristic function of a smooth bounded domain this domain is uniquely determined by the scattering data.*

A potential satisfying the assumptions of the last Corollary is in $H^s_{comp}(R^2)$ for every $s < 1/2$. Thus by Theorem 1, $q_B(x) - q(x)$ is continuous and by Theorem 2, $q_B^b(x) - q(x)$ is in $H^t(R^2)$ for every $t < 3/4$. Hence this Corollary is satisfied.

Theorem 3 *Assume that the potential $q(x)$ belongs to the space $L^p(R^2)$ with $3/2 < p < 5/2$ and has the special behaivior (2) at the infinity. Then*

$$q_B^{\vartheta_0}(x) - q(x) \in H^t(R^2),$$

where $t < 1/2 - 3/4p$.

This theorem means that the leading order singularities of the unknown potential can be obtained exactly by the Born approximation (7) with fixed angle.

Theorem 4 *Assume that the potential $q(x)$ belongs to the space $L^p(R^2)$ for $3/2 < p \leq \infty$ and has the special behaivior (2) at the infinity. Then*

$$q_B^f(x) - q(x) - q_1(x)$$

is a continuous function, where the first nonlinear term $q_1(x)$ from the Born expansion belongs to $W^1_{\frac{2p}{4-p},loc}(R^2)$ for $3/2 < p < 2$ and belongs to $W^1_{r,loc}(R^2)$ with arbitrary $r < 2$ for $p = 2$ and $q_1(x)$ is bounded and continuous for $2 < p \leq \infty$.

Theorem 5 *Assume that the potential $q(x)$ belongs to the space $L^p(R^2)$ for $4/3 < p \leq 3/2$ and has the special behaivior (2) at the infinity. Then*

$$q_B^f(x) - q(x) - q_1(x) \in H^t(R^2)$$

where $t < 3 - 4/p$ and the first nonlinear term $q_1(x)$ from the Born expansion belongs to $W^1_{\frac{2p}{4-p},loc}(R^2)$.

The embedding theorems for Sobolev spaces lead to the fact that the first nonlinear term $q_1(x)$ from the Born expansion which corresponds to the fixed energy inverse problem actually belongs to $L_{loc}^{\frac{p}{2-p}}(R^2)$ if $3/2 < p < 2$ and to $L^s_{loc}(R^2)$ for any $s < 2$ if $p = 2$. This and Theorems 4 and 5 give that the leading order singularities of the unknown potential can be obtained exactly by the Born approximation. Concerning the case $2 < p \leq \infty$ Theorem 4 gives that all singularities and jumps of the unknown potential can be obtained exactly by the Born approximation. In particular, if the potential is the characteristic function of a bounded domain this domain is uniquely determined by the fixed energy scattering data.

Note also that in the case of compactly supported potential $q(x)$ the Dirichlet to Neumann map Λ_q uniquely determines T_q (see [25], [26], [27], [10] and [12]). Hence, the question of determining the singularities of q from Λ_q or from the scattering amplitude at fixed positive energy can be reduced to determining singularities of q from T_q. Thus we get

Theorem 6 *Let $\Omega \subset R^2$ be a smooth bounded domain. Assume that $q_1, q_2 \in L^p(\Omega), 4/3 < p \leq \infty, \Lambda_{q_1} = \Lambda_{q_2}$, and that 0 is not a Dirichlet eigenvalue of Ω. Then*

$$q_1(x) = q_2(x) \quad mod(W^1_{r,loc}(R^2) + H^t_{loc}(R^2))$$

where $r = \frac{2p}{4-p}$ for $4/3 < p < 2$, and with arbitrary $r < 2$ for $p = 2$, and $t < 3 - 4/p$ for $4/3 < p \leq 3/2$, and $t < 2 - 3/2p$ for $3/2 < p \leq 2$, and

$$q_1(x) = q_2(x) \quad (C_{loc}(R^2))$$

for $2 < p \leq \infty$.

Theorem 7 (Saito's formula) *Under the same assumptions for $q(x)$ as in Theorem 4 or in Theorem 5*

$$\lim_{k\to+\infty} k \int\limits_{S^1\times S^1} e^{-ik(\vartheta-\vartheta',x)} A(k,\vartheta',\vartheta)\, d\vartheta\, d\vartheta' = 4\pi \int\limits_{R^2} \frac{q(y)}{|x-y|}\, dy.$$

This limit is valid in the sense of distributions.

The next theorem is the uniqueness theorem of the reconstruction of unknown potential by the scattering amplitude and it is a simple corollary from Saito's formula.

Theorem 8 *Assume that the potentials $q_1(x)$ and $q_2(x)$ satisfy the conditions of Theorem 7 and the corresponding scattering amplitudes coincide for some sequence $k_j \to +\infty$ and for all $\vartheta, \vartheta' \in S^1$. Then $q_1(x) = q_2(x)$ (in the sense of distributions).*

It follows from the Saito's formula a very interesting connection between the potential $q(x)$ and the scattering amplitude $A(k, \vartheta', \vartheta)$

$$q(x) = \lim_{k \to +\infty} \frac{k^2}{8\pi^2} \int\limits_{S^1 \times S^1} e^{-ik(\vartheta - \vartheta', x)} A(k, \vartheta', \vartheta) |\vartheta - \vartheta'| \, d\vartheta \, d\vartheta'.$$

This formula should be understood also in the sense of distributions.

It follows from the proof of Saito's formula in the case when we have the scattering amplitude only with one fixed direction ϑ_0 that

$$\lim_{k \to +\infty} k^{1/2} \int\limits_{S^1} e^{-ik(\vartheta - \vartheta_0, x)} A(k, \vartheta_0, \vartheta) \, d\vartheta = 0.$$

This limit is valid uniformly with respect to x.

The new asymptotical formula for the unknown potential which contains only the Green's function of the Hamiltonian is presented in following theorem.

Theorem 9 *Assume that the potential $q(x)$ belongs to $L^p(R^2)$ for some $1 < p \leq \infty$ and has the special behaviour at the infinity (2). Then the Fourier transform $F(q)$ of the potential $q(x)$ belongs to the space $L^\infty(R^2) \cap C(R^2)$ and at every point ξ can be calculated by the formula*

$$F(q)(\xi) = \lim_{|x|, |y|, k \to +\infty} 8\pi i k (|x||y|)^{1/2} e^{-ik(|x|+|y|)} (G_q(x, y, k) - G_k^+(|x - y|)),$$

where $\xi = -k(\frac{x}{|x|} + \frac{y}{|y|})$ and the function G_q satisfies the integral equation (4).

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